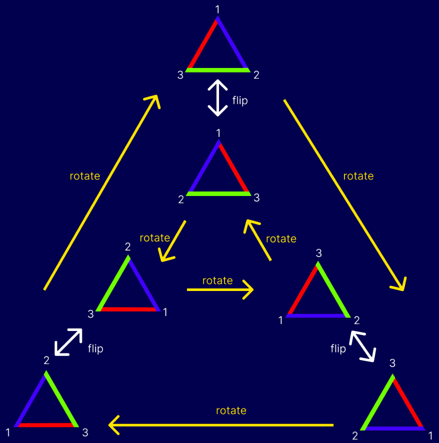
Finite Group Theory: the major problems, and why we care



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A *group* is a pair $(G, \circ)$ where G is a set, and $\circ : G \times G \to G$ is a binary operation on G that satisfies the following three properties:

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(ii) **Identity**. G has an element, which we call 1_G, satisfying $g \circ 1_G = 1_G \circ g = g$ for all $g \in G$. We call 1_G an *identity element* in $(G, \circ)$.

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Most branches of algebra we study today (i.e. Linear Algebra; Ring Theory; Group Theory; Module Theory, etc.) were built from a desire to get rigorous answers to questions from other areas of mathematics and science.

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Let's look back at the definition:

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Part (i) represents the fact that, if I perform two symmetries (g and h) and then perform another (k) some time later; this is the same as performing g, waiting a while, then performing h and k.

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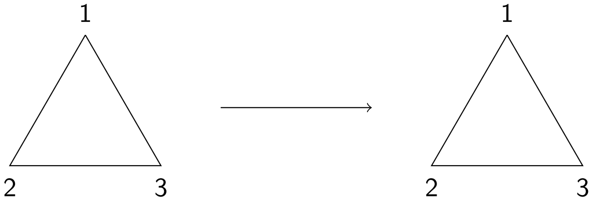
In part (ii), the identity represents the "do nothing" symmetry of an object.



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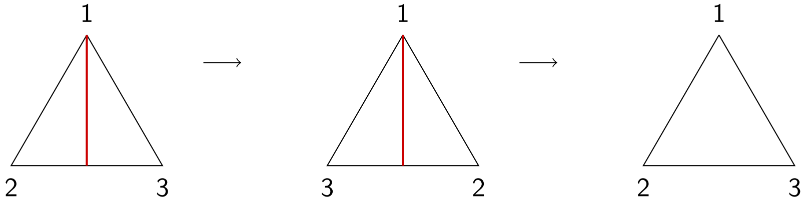
Part (iii) encodes the fact the "reverse" of every symmetry is still a symmetry.



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Historical note: The first definition of a group was given by Galois in 1830, and it was less abstract than the one above. Indeed, Galois defined *a group of substitutions of degree n* to be what we now call a subgroup of the symmetric group S_n. This shows that the study of groups is fundamentally motivated by the desire to understand symmetry.

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Because symmetry is so universal, Group Theory is highly ubiquitous: it arises naturally not only in many fundamental areas of mathematics (like Geometry, Topology, Number Theory, Harmonic Analysis and more); but also in other areas of human study (like Virology, Chemistry, Physics, Computer Science, Cryptography..).

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The Extension Problem

Classify all of the finite groups.

For various other values of n (always with hard restrictions on the prime divisors of n), the finite groups of order n have been classified.

In particular, we know the finite groups of order up as far as $2047 = 2^{11} - 1$. Here are the number of groups of order 2^k, for $k \leq 10$.

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We will come back to this heuristic later in the talk, but for now...

Moral of the story: In general, the Extension Problem is intractable! For this reason, group theorists focus on other questions/areas, which aim to get us as close to a solution to the Extension Problem as possible. For the remainder of the talk, I will speak about three of these questions/areas:

1. The Classification of Finite Simple Groups;
2. Asymptotic group theory;
3. Burnside's problems.

1. The Classification of Finite Simple Groups

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But why are the finite simple groups so important?

The Jordan–Hölder theorem

Theorem (Jordan–Hölder theorem)

Every finite group G has a composition series, i.e. a series

$$\{1_G\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$$

such that for each $1 \leq i \leq r$, the group G_i/G_{i-1} is simple. Moreover, although there can be different composition series, the length r, and the isomorphism classes of the factors G_i/G_{i-1} do not change. Thus, the multiset $\{\{G_i/G_{i-1} : 1 \leq i \leq r\}\}$ is well-defined, and is called the set of composition factors for G.

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E.g. The alternating groups A_n are simple for $n \geq 5$, while the cyclic groups of prime order are simple. The composition factors of S_n for $n \geq 5$ are $\{\{A_n, C_2\}\}$.

E.g. For p an odd prime, the composition factors of the dihedral group D_{2p} of order $2p$ are $\{\{C_p, C_2\}\}$.

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Theorem (Burnside's $p^a q^b$ theorem)

Let p and q be primes, and let G be a group of order $p^a q^b$. Then G is not simple.

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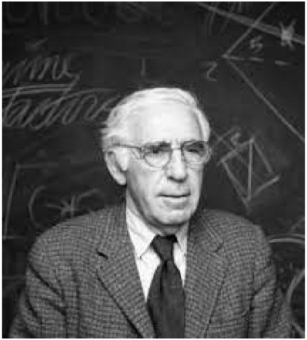
Encouraged by this, together with some other deep results on finite simple groups like the Brauer-Fowler theorem, Daniel Gorenstein announced in 1972 an ambitious 16 step programme that would classify the finite simple groups.



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Side note: The last category is a set of finite simple groups of order between 7920 (the order of the *Mathieu group* M_{11}) and

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These groups do not fit naturally into any of the preceding three infinite families.

Gorenstein's programme

In pursuit of a contradiction, one then chooses a finite simple group G such that G is not one of those listed above, and $|G|$ is as small as possible.

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The minimality of $|G|$ also shows that all subgroups of G have their composition factors lying in our know list of finite simple groups..

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"Mathematicians theorize themselves out of a job"

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Why are the New York Times wrong?

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Also, the finite simple groups on our list are complicated! We still don't know everything we need to know about them (for example, we still haven't been able to classify all of their maximal subgroups). This brings us nicely to..

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The final proof will be more uniform in approach, and will be shorter. (The plan is a volume of 11 books, comprising about 3000 pages in total. Book 9 is almost ready..).

2. Asymptotic group theory

In this talk, we've focused mostly on deterministic type problems in finite group theory. That is, problems of the form: "Classify the finite groups with property $\mathcal{P}$..".

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The philosophy behind asymptotic group theory is to say: "OK, we can't classify finite the finite groups with property $\mathcal{P}$, but can we say something about how many groups satisfy property $\mathcal{P}$? Or if we choose a finite group *at random* from a certain list, then how likely it is that the group we choose satisfies property $\mathcal{P}$?

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Both of these conjectures are still open.

The number of isomorphism classes of finite groups of order at most n

For a positive integer n with prime factorisation

$$n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} \ (p_i \text{ distinct primes})$$

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In 1990, Pyber proved the following incredible result.

Theorem (Pyber, 1990)

Let $n \in \mathbb{N}$ and let $\mu := \mu(n)$. We have

$$f^*(n) \leq n^{\mu^2/27 + h(\mu)}$$

where $h(\mu) = o(\mu^2)$.

By work of Higman and Sims from the 1960s, the bound in the above theorem is "best possible".

More on Pyber's theorem

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He conjectured, however, that much more is true. (The following would be best possible).

Conjecture (Pyber, 1990)

Let $n \in \mathbb{N}$. The number of subgroups of the symmetric group S_n is at most $2^{n^2/16+h(n)}$, where $h(n) = o(n^2)$.

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Theorem (Roney-Dougal & T., 2023)

Pyber's conjecture holds. In fact, the number of subgroups of the symmetric group S_n is at most $2^{n^2/16+cn^{3/2}}$, where c is some absolute constant.

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Recall the probabilistic conjectures of Erdös and Pyber from earlier.

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An analogue for permutation groups was proposed by Kantor.

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Theorem (Roney-Dougal & T., 2024)

Kantor's conjecture is not true. Indeed, there exists an absolute constant $\epsilon > 0$ such that $|\mathrm{Sub}_2(S_n)|/|\mathrm{Sub}(S_n)| < 1 - \epsilon$ for all $n \in \mathbb{N}$.

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One can take $\epsilon = 1/2^{16^2}$.

A number of questions arise from this theorem:

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The final part of Question 2 is not "just for the sake of it". There is very practical motivation..

The Group Isomorphism Problem

The *Group Isomorphism Problem* (henceforth abbreviated to GrpI) is the decision problem for determining whether or not two groups G_1 and G_2 given by their multiplication tables are isomorphic.

The Group Isomorphism Problem (version we usually study)

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What has this got to with counting subgroups of finite groups?!

An intriguing new approach due to Gowers shows that an improved understanding of subgroup enumeration could lead to remarkable progress in GrpI.

The idea is as follows: For finite groups G and X, let $\mathrm{Sub}_{\cong X}(G)$ be the set of subgroups H of G with $H \cong X$. For $n \in \mathbb{N}$, let $k(n)$ be the smallest positive integer such that for any two finite groups G_1 and G_2 of order n, we have $G_1 \cong G_2$ if and only if $|\mathrm{Sub}_{\cong X}(G_1)| = |\mathrm{Sub}_{\cong X}(G_2)|$ for all $k(n)$-generated finite groups X. Then the GrpI can be solved in time $n^{k(n)}$.

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All we know at the moment is that $k(n) \leq \log n$..

3. Burnside's problems

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In 1901, Burnside asked the following famous question.

Burnside's problem

Let G be a group which can be generated by d elements (G is said to be *d-generated* in this case), and such that $g^n = 1_G$ for all $g \in G$ (G is said to have *exponent* n in this case). Is G necessarily finite?

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Definition

Let G be a group, and let A be a non-empty subset of G. The *subgroup of G generated by A*, written $\langle A \rangle$, is defined to be

$$\langle A \rangle := \{a_1^{\epsilon_1} \ldots a_m^{\epsilon_m} : m \in \mathbb{N}, a_i \in A, \epsilon_i \in \mathbb{Z}\}.$$

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So Burnside's problems says that if G is a group, there exists $A \subseteq G$ of size d such that every element can be written as a product of powers of elements of A, and every element of G has order at most n, then is G finite? Or, more informally, if G "operationally" finite, then is G finite?

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Example 1 Suppose that G, d and n are as in Burnside's question, and $n = 2$ (i.e. G is a d-generated group in which $g^2 = 1_G$ for all $g \in G$).

Then $g = g^{-1}$ for all $g \in G$, so we have

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Thus, G is abelian. Hence, writing $A = \{x_1, \ldots, x_d\}$, we have

$$\begin{aligned} |G| = |\langle A \rangle| &= |\{a_1^{\epsilon_1} \ldots a_m^{\epsilon_m} : m \in \mathbb{N}, a_i \in A, \epsilon_i \in \mathbb{Z}\}| \\ &= |\{x_1^{e_1} \ldots x_d^{e_d} : e_i \in \{0, 1\}\}| \leq 2^d. \end{aligned}$$

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The same approach won't work (there are examples of finite groups of exponent 3 which are not abelian). But is there anything we can do?

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Proof of claim: Let G be as in the statement, and let a and b be elements of G. Then we have

$$\begin{aligned} 1_G = (ba)^3 &= bababa \\ &= b(b^2b^{-2})a(b^2b^{-2})baba \\ &= b^3(b^{-2}ab^2)(b^{-2}b)aba \\ &= (bab^{-1})(b^{-1}ab)a \qquad \text{since } b^3 = 1_G\text{, and hence } b = b^{-2}. \\ &= (bab^{-1})(b^2ab^{-2})a. \end{aligned}$$

We therefore have $a^{-1} = a^{b^{-1}}a^b$. Replacing b by b^{-1} gives $a^{-1} = a^b a^{b^{-1}}$. Thus,

$$a^{b^{-1}}a^b = a^b a^{b^{-1}} \text{ for all } a, b \in G.$$

So raising both sides to the b^2 yields:

$$a^b a = aa^b \text{ for all } a, b \in G.$$

The point of the previous claim is that if G is a group of exponent 3 which can generated by d elements (say $a_1, \ldots, a_d$), then the group

$$N := \langle a_d^b : b \in G \rangle$$

is abelian, hence isomorphic to C_3^m for some m (by the same argument as used in Example 1). Since $N \trianglelefteq G$ and G/N can be generated by $\langle a_1 N, \ldots, a_{d-1} N \rangle$, an easy inductive argument shows that G is finite.

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NOBODY KNOWS..

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   These are called *Tarski Monsters*, and they exist for every prime $p > 10^{75}$.

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The Restricted Burnside problem

With the Hall–Higman reduction in mind, group theorists began working furiously on the restricted Burnside problem in prime power exponent. This allows one to assume that the group G one is working in is a finite p-group, for some prime p (i.e. G has p-power order).

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The first significant breakthrough came from Kostrikin in 1959.

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Fix $d \in \mathbb{N}$ and a prime p. There are only finitely many finite d-generated groups G of exponent p. That is, the Restricted Burnside problem has a positive solution for prime exponents.

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Kostrikin's idea used an intriguing connection between finite groups of p-power order, and *Lie algebras* (on which Warwick's Adam Thomas is one of the world's foremost experts!)

Lie algebras and finite p-groups

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(d) The *Jacobi identity* $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$ holds for all $u, v, w \in L$.

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(b) $[\cdot,\cdot] : L \times L \to L$ is a bilinear map. That is, it satisfies $[v+w, u] = [v, u] + [w, u]$ and $[\lambda v, u] = \lambda[v, u]$ for all $\lambda \in \mathbb{F}$; and also the analogous linearity relations in the second coordinate.

(c) $[v, v] = 0$ for all $v \in L$.

(d) The *Jacobi identity* $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$ holds for all $u, v, w \in L$.

The standard example is $L := M_n(\mathbb{F})$, the set of $(n \times n)$-matrices over $\mathbb{F}$, where $[A, B] := AB - BA$.

Lie algebras and finite p-groups

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The connection

Let G be a finite group of exponent p (i.e. $g^p = 1_G$ for all $g \in G$). The *commutator subgroup* $\gamma_2(G) := [G, G]$ is defined by

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Since $G/[G, G]$ is abelian, and G has exponent p, we have that $G/[G, G] \cong (\mathbb{Z}/p\mathbb{Z})^n$. That is, $G/[G, G]$ is a vector space (of dimension n) over the field $\mathbb{F} := \mathbb{Z}/p\mathbb{Z}$.

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We now define $\gamma_1(G) := G$, and for $i \geq 2$,

$$\gamma_i(G) := \langle [x, g] : x \in \gamma_{i-1}(G), g \in G \rangle.$$

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$$[v\gamma_i(G), w\gamma_j(G)] := [v, w]\gamma_{i+j \pmod{n}}(G).$$

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In much the same way that Galois theory uses Group Theory to solve problems in Number Theory, Kostrikin used this connection between finite p-groups of exponent p and Lie algebras over $\mathbb{F} := \mathbb{Z}/p\mathbb{Z}$ to make the extraordinary breakthrough on the Restricted Burnside problem mentioned above.

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The case of general prime power exponent $d := p^m$ was done by Zelmanov in the late 1980s. He used a similar connection to the above, but needed to work with *Lie rings* rather than Lie algebras, as $\mathbb{Z}/p^m\mathbb{Z}$ is a field if and only if $m = 1$.

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In 1994, Zelmanov was awarded the Fields medal for his work.

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Some see the beauty of Group Theory in how fundamental it is to so many different areas of mathematics and science.

In my view, another huge aspect of its beauty is how quickly one can get from the definition of a group, to deep and important problems.

For example, if you can prove that a 2-generated group G in which $g^5 = 1_G$ for all $g \in G$, is finite, then you will (without a doubt) win a Fields medal...